Knot soliton models, submodels, and their symmetries

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Abstract

For some non-linear field theories which allow for soliton solutions, submodels with infinitely many conservation laws can be defined. Here we investigate the symmetries of the submodels, where in some cases we find a symmetry enhancement for the submodels, whereas in others we do not.

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1 Introduction

Non-linear field theories with a two-dimensional target space and base space $\mathbb{R} \times \mathbb{R}^d$ ($d+1$ dimensional space-time) can give rise to point like (vortex like) solitons for $d = 2$, or to line like (knot like) solitons for $d = 3$, provided that the fields are required to approach a fixed, constant value at spatial infinity (e.g., to render the energy finite), compactifying thereby the base space $\mathbb{R}^d$. Especially, for $d = 3$ some models with knot solitons have received considerable attention recently and, further, such models have applications both in condensed matter [1, 2] and elementary particle physics [3, 4]. Here we concretely consider models where the target space is the two-sphere S^2. Their solitons can be classified by the homotopy groups $\pi_2(S^2) = \mathbb{Z}$ (winding number, for vortex type solitons) and $\pi_3(S^2) = \mathbb{Z}$ (Hopf index, for knot type solitons), respectively.

The fields of the theories may be parametrized by a three-component unit vector $\mathbf{n} : \mathbb{R} \times \mathbb{R}^d \to S^2$, $\mathbf{n}^2 = 1$, or via the stereographic projection

$$\mathbf{n} = \frac{1}{1 + u\bar{u}} \left(u + \bar{u}, -i(u - \bar{u}), u\bar{u} - 1 \right), \quad u = \frac{n_1 + in_2}{1 - n_3} \tag{1}$$

by a complex scalar field u.

All models which we study can be constructed from the two Lagrangian densities

$$\mathcal{L}_2 = \frac{\partial_\mu u \, \partial^\mu \bar{u}}{(1 + u\bar{u})^2} \tag{2}$$

and

$$\mathcal{L}_4 = \frac{(\partial^\mu u \, \partial_\mu \bar{u})^2 - (\partial^\mu u \, \partial_\mu u)(\partial^\nu \bar{u} \, \partial_\nu \bar{u})}{(1 + u\bar{u})^4}. \tag{3}$$

In two space dimensions we consider the Baby Skyrme model $\mathcal{L}_{\mathrm{BS}} = \mathcal{L}_2$, whereas in three space dimensions we will consider the Faddeev–Niemi model [5, 6] with Lagrangian

$$\mathcal{L}_{\mathrm{FN}} = \mathcal{L}_2 - \lambda \mathcal{L}_4 \tag{4}$$

(here λ is a dimensionful coupling constant), the Nicole model

$$\mathcal{L}_{\mathrm{Ni}} = (\mathcal{L}_2)^{\frac{3}{2}} \tag{5}$$

(for which the one known soliton solution was found by Nicole, [7]), and the AFZ model

$$\mathcal{L}_{\mathrm{AFZ}} = -(\mathcal{L}_4)^{\frac{3}{4}}, \tag{6}$$

for which infinitely many soliton solutions have been found by Aratyn, Ferreira and Zimerman (=AFZ) [8, 9]. All four models circumvent Derrick's theorem and allow for static soliton solutions, either by being spatially scale invariant (the Baby Skyrme, the Nicole, and the AFZ model), or by involving two terms with opposite scaling behaviour (the Faddeev–Niemi model).

All four models (Baby Skyrme, Faddeev–Niemi, AFZ and Nicole) have the same target space S^2, therefore they have some common properties. For instance, all Lagrangians are invariant under modular transformations

$$u \;\rightarrow\; \frac{au+b}{-\bar{b}u+\bar{a}}\,, \quad a\bar{a}+b\bar{b}=1. \tag{7}$$

Furthermore, the same area-preserving diffeomorphisms on the target space S^2 can be defined for all models, but this does not imply that they are symmetries for all four field theories. In fact, only the AFZ model has the area-preserving diffeomorphisms as symmetries [10, 11]. For the other three models the generators of the area-preserving diffeomorphisms do not generate symmetries and the corresponding Noether currents are not conserved. However, it was realized in the study of higher-dimensional integrability within the generalization of the zero curvature representation, [12], that these Noether currents *are* conserved for submodels of all three models defined by the additional condition

$$\partial^\mu u \partial_\mu u = 0, \tag{8}$$

i.e., the complex eikonal equation. Therefore, these submodels have infinitely many conserved charges. On the other hand, their symmetries have to be determined independently, because the complex eikonal equation is not of the Euler–Lagrange type, i.e., it does not follow from an action, and the Noether theorem does not apply to the submodels. This symmetry determination is the main purpose of our talk.

In Section 2 we give a very brief survey of the issue of integrability in higher dimensions and of the resulting infinitely many conservation laws (i.e., conserved currents). In Section 3 we introduce a general class of Lagrangians (to which, of course, all models mentioned above belong) which provide a Lagrangian realization of the infinitely many conserved currents of Section 2. Further, we explain the geometric significance of these currents and their conservation. In Section 4 we briefly investigate the symmetries of the static

equations of motion (which are the relevant ones for solitons) for the three submodels (of the Baby Skyrme, Faddeev–Niemi and Nicole models). Section 5 contains our conclusions.

2 Brief survey of integrability in higher dimensions

In [12] a generalization of the zero curvature condition of Zakharov and Shabat in 1+1 dimensional integrable models was introduced in order to extend the concept of integrability to field theories in higher dimensions. In its original formulation, this condition was a zero curvature in a generalized loop space which leads to very non-local expressions when re-expressed in terms of fields over ordinary space-time. In the same paper, however, a *local* condition realizing this generalized zero curvature condition was given, which we want to describe briefly here. We choose a non-semisimple Lie algebra $\tilde{\mathcal{G}}$ which is the direct sum of a (possibly, but not necessarily semi-simple) Lie algebra $\mathcal{G}$ and an abelian ideal $\mathcal{P}$, i.e.,

$$\tilde{\mathcal{G}} = \mathcal{G} + \mathcal{P} \tag{9}$$

where $\mathcal{P}$ may, e.g., be a (in general, reducible) representation of $\mathcal{G}$, in which case we have

$$\begin{aligned} [T^a, T^b] = f^{abc} T^c \quad &, \quad [T^a, P^n] = R^{mn}(T^a) P^m \\ [P^m, P^n] = 0 \quad &, \quad T^a \in \mathcal{G}\,, \quad P^m \in \mathcal{P} \end{aligned} \tag{10}$$

and $R^{mn}(T^a)$ are matrices in the representation $\mathcal{P}$. Further, we choose a flat connection $A_\mu \in \mathcal{G}$, i.e.,

$$\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0 \quad \Rightarrow \quad A_\mu = g^{-1}\partial_\mu g \tag{11}$$

where $g \in \mathbf{G}$ and $\mathbf{G}$ is the Lie group of which $\mathcal{G}$ is the Lie algebra. Finally, we need a covariantly conserved, vector-valued element of the abelian ideal, $B_\mu \in \mathcal{P}$, i.e.,

$$\partial^\mu B_\mu + [A^\mu, B_\mu] = 0, \tag{12}$$

then there exist the conserved currents

$$J_\mu \equiv g B_\mu g^{-1}\,, \qquad \partial^\mu J_\mu = 0 \tag{13}$$

as may be checked easily. If this construction holds for $\dim(\mathcal{P}) = \infty$ then we have infinitly many conserved currents.

For our purposes we now specialize to $\mathbf{G} = SU(2)$ and choose as the group element g a fixed, given function of the field $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ and its complex conjugate,

$$g = g(u, \bar{u}) \in SU(2). \tag{14}$$

Essentially, g takes values on the equatorial two-sphere contained within $SU(2)$ when u takes values in $\mathbb{C}$ (for an explicit expression see [12]). The representations P^m are now just the standard representations $P^{(l,m)}$ of $SU(2)$ where l and m are the angular momentum and magnetic quantum numbers, respectively. Further we restrict to $m = \pm 1$, i.e.,

$$B_\mu = \sum_l c_l B_\mu^l, \tag{15}$$

$$B_\mu^l = K_\mu P^{(l,1)} + \bar{K}_\mu P^{(l,-1)} \tag{16}$$

where the c_l are arbitrary real constants (making the abelian ideal infinite-dimensional), and $K_\mu(u, \bar{u}, u_\mu, \bar{u}_\mu)$ is a given function of the field variables $u, \bar{u}$ and its first derivatives (in principle, also of higher derivatives, but we do not consider this possibility here). Here and below we use the notation $\partial_\mu u \equiv u_\mu$. Different choices for K_μ correspond to different field theories, as we shall see. Further, K_μ has to obey the reality condition

$$\Im(\bar{u}_\mu K^\mu) = 0. \tag{17}$$

For the so chosen B_μ^l, the corresponding currents $J_\mu^l = g B_\mu^l g^{-1}$ are equivalent to the currents

$$J_\mu^G = i(K_\mu G_u - \bar{K}_\mu G_{\bar{u}}) \tag{18}$$

for an arbitrary *real* function $G(u, \bar{u})$ ($G_u \equiv \partial_u G$), see [10]. If all J_μ^l are conserved, then J_μ^G is conserved for arbitrary functions G. In the next Section, we shall present a Lagrangian realization of these integrability concepts.

3 Lagrangian realization of conserved currents

We introduce the class of Lagrangian densities

$$\mathcal{L}(u, \bar{u}, u_\mu, \bar{u}_\mu) = F(a, b, c) \tag{19}$$

where

$$a = u\bar{u}\,, \quad b = u_\mu \bar{u}^\mu\,, \quad c = (u_\mu \bar{u}^\mu)^2 - u_\mu^2 \bar{u}_\nu^2 \tag{20}$$

and F is at this moment an arbitrary real function of its arguments. The phase symmetry $u \to e^{i\alpha} u$ for a constant $\alpha \in \mathbb{R}$ holds by construction. For the vector-valued function K^μ we choose

$$K^\mu = f(a)\bar{\Pi}^\mu \tag{21}$$

where f is a real function of its argument, and Π^μ and $\bar{\Pi}^\mu$ are the conjugate four-momenta of u and $\bar{u}$, i.e. ($u_\mu \equiv \partial_\mu u$, $F_b \equiv \partial_b F$, etc.),

$$\Pi_\mu \equiv \mathcal{L}_{u^\mu} = \bar{u}^\mu F_b + 2(u^\lambda \bar{u}_\lambda \bar{u}_\mu - \bar{u}_\lambda^2 u_\mu) F_c. \tag{22}$$

K^μ in Eq. (21) automatically obeys the reality condition (17) for real Lagrangian densities. For the divergence $\partial^\mu J_\mu^G$ of the current (18) we find

$$\begin{aligned} \partial^\mu J_\mu^G &= if \Big([(M'\bar{u}G_u + G_{uu})u_\mu^2 - (M'uG_{\bar{u}} + G_{\bar{u}\bar{u}})\bar{u}_\mu^2] F_b \\ &\quad + (uG_u - \bar{u}G_{\bar{u}})[M'(bF_b + 2cF_c) + F_a] \Big) \end{aligned} \tag{23}$$

where $M \equiv \ln f$, $M' \equiv \partial_a M$, and we used the equations of motion

$$\partial^\mu \Pi_\mu = \mathcal{L}_u = \bar{u} F_a. \tag{24}$$

Now we want to study under which circumstances the divergence (23) vanishes (for a more detailed discussion we refer to Ref. [13]). If no constraints are imposed neither on the Lagrangian nor on the allowed class of fields u, then we find the two equations for G,

$$uG_u - \bar{u}G_{\bar{u}} = 0, \tag{25}$$

and

$$M_a \bar{u} G_u + G_{uu} = 0 \quad \Rightarrow \quad \partial_u [f(u\bar{u})G_u] = 0, \tag{26}$$

with the solution

$$G_u = k\frac{\bar{u}}{f} \tag{27}$$

where k is a real constant. The corresponding current J_μ^G is the Noether current for the phase transformation $u \to e^{i\alpha} u$ which is a symmetry by construction.

Next we make the second term in (23) vanish by imposing on the Lagrangian the condition

$$M_a(bF_b + 2cF_c) + F_a = 0 \tag{28}$$

with the general solution

$$F(a,b,c) = \tilde{F}(\frac{b}{f}, \frac{c}{f^2}) \tag{29}$$

which has an interpretation in terms of the target space geometry. In fact, introduce the two real target space coordinates ξ^α via $u = \xi^1 + i\xi^2$ and the target space metric w.r.t. to the coordinates ξ^α via

$$g_{\alpha\beta} \equiv f^{-1}\delta_{\alpha\beta} \quad \Rightarrow \quad \det(g_{\alpha\beta}) \equiv f^{-2} \tag{30}$$

$$\tilde{\epsilon}_{\alpha\beta} = f^{-1}\,\epsilon_{\alpha\beta} \tag{31}$$

where $\epsilon_{\alpha\beta}$ is the usual antisymmetric symbol in two dimensions. Then the expressions on which $\tilde{F}$ may depend can be written as

$$\frac{b}{f} = g_{\alpha\beta}(\xi)\partial^\mu\xi^\alpha\partial_\mu\xi^\beta \tag{32}$$

$$\frac{c}{f^2} = \tilde{\epsilon}_{\alpha\beta}\tilde{\epsilon}_{\gamma\delta}\partial^\mu\xi^\alpha\partial_\mu\xi^\gamma\partial^\nu\xi^\beta\partial_\nu\xi^\delta \tag{33}$$

i.e., they depend on the target space metric and on the determinant of the target space metric, respectively. Let us point out here that all models of Section 1 are of this type, i.e., $\mathcal{L}_2 = b/f$ and $\mathcal{L}_4 = c/f^2$ for $f = (1+a)^2$ (the target space metric of the two-sphere).

To make the first term in Eq. (23) vanish, as well, we may either continue to impose Eq. (26), which is solved by those G which generate the target space isometries for the given target space metric (i.e., the given function f). Or we may restrict the Lagrangian further by imposing $F_b \equiv 0 \Rightarrow F = \tilde{F}(c/f^2)$. Then we have no restriction on G at all, and it follows that these unrestricted G generate the area-preserving diffeomorphisms on target space. This is precisely the case for the AFZ model.

Alternatively, we may make the first term in Eq. (23) vanish by imposing restrictions on the allowed field configurations u. In this case the currents J^G_μ are still the Noether currents of area-preserving diffeomorphisms, but these

transformations are no longer symmetry transformations of the pertinent Lagrangians, in general. Concretely, we require that u obeys the complex eikonal equation (8), which defines therefore submodels for all models of the type (29) with infinitely many conserved charges.

[Remark: we might require, instead, that the field u obeys the (in general nonlinear) first order PDE which follows from the condition $F_b = 0$ in cases when this condition does not hold identically (i.e., for Lagrangians which do depend on the term $b = u^\mu \bar{u}_\mu$). This type of ("generalized") integrability condition, which depends, however, on the chosen Lagrangian, has been discussed in [14], [13].]

4 Symmetries of the static equations

Here we just want to present the results of the calculation of all geometric symmetries (point symmetries) of the static equations, i.e., the static equations of motion (e.o.m.) for the full models (Baby Skyrme, Nicole and Faddeev–Niemi), and the static equations of motion plus the static eikonal equation for the corresponding submodels (observe that the static complex eikonal equation does have nontrivial solutions, in contrast to its real counterpart, see, e.g., [15]). The method of prolongations has been used for all symmetry calculations. Concretely, the symmetries of the submodels are calculated by first calculating the on-shell symmetries of the static eikonal equation. In a next step the on-shell symmetries of the static second order equations are calculated, where the second order equations consist of the equations of motion plus the prolongations of the static eikonal equation (i.e., the second order equations that follow by acting with total derivatives on the complex eikonal equation). For the calculations we refer to [16]. Here we give a detailed discussion of the results, which are displayed in Table 1.

For the Baby Skyrme model we find that the full static model has a point symmetry group which is a direct product of base space and target space symmetries, where the group of base space symmetries is the conformal group in two dimensions C_2, and the group of target space symmetries is the group $SU(2)$. On the other hand, the submodel has the point symmetry group $C_2 \times C_2$, i.e., the conformal group also in target space. Therefore, the submodel has more symmetry in the case of the Baby Skyrme model, although the additional symmetry is not related to the area-preserving diffeomorphisms.

model	∞ many conserv. laws	geometric symmetries	solutions known
Baby Skyrme	yes[a]	$C_2 \times SU(2)$	yes
submodel	yes	$C_2 \times C_2$	yes
Nicole	no	$C_3 \times SU(2)$	yes
submodel	yes	$C_3 \times SU(2)$	yes
Faddeev–Niemi	no	$E_3 \times SU(2)$	yes[b]
submodel	yes	$E_3 \times SU(2)$	no

Table 1: Some results for the three soliton models and their submodels.
C_d ... conformal group in d dimensions.
E_d ... Euclidean group (translations and rotations) in d dimensions.
[a]due to the infinite-dimensional base space symmetries C_2.
[b]known only numerically

Further, there exist static solutions to the submodel. In fact, *all* soliton solutions of the Baby Skyrme model are, at the same time, also solutions of the submodel.

For the Nicole model we find that the group of point symmetries of the static e.o.m. is again a direct product of base space and target space symmetry groups, where the base space symmetry group is C_3, the conformal group in three dimensions, and the target space symmetry group is $SU(2)$. Further, the static submodel has exactly the same symmetry group $C_3 \times SU(2)$ as the full Nicole model. For the Nicole model only one analytical soliton solution is known, but this solution solves the static eikonal equation, as well, and is, therefore, also a solution of the submodel [7, 16].

For the Faddeev–Niemi model the situation is similar. Again, the static submodel has exactly the same symmetries as the full static model, and the symmetry group is a direct product of the Euclidean group in three dimensions E_3 in base space (i.e., rotations and translations) and of the group $SU(2)$ in target space. For the full Faddeev–Niemi model soliton solutions are known only numerically [3], [17] - [20]. It is not known whether the submodel does or does not have solutions.

5 Conclusions

In this talk we gave a brief survey of the generalized zero curvature representation of [12], which leads to a generalization of integrability to higher-dimensional non-linear field theories. Then we introduced a class of Lagrangian field theories parametrized by a complex field variable u, where this concept of integrability is realized by providing infinitely many conserved currents either for the full theory or for the submodel defined by the eikonal equation $(\partial u)^2 = 0$. Further, these currents may be interpreted as Noether currents of area-preserving diffeomorphisms on the target space where u takes its values. For some relevant models within this class of field theories, which allow for soliton solutions, we then presented the results of a thorough analysis of the symmetries of their submodels.

The general result for all cases is that the area-preserving diffeomorphisms are not symmetries of any eikonal submodel. Also, the three-dimensional submodels of Faddeev–Niemi and Nicole have no additional symmetries compared to the full theories.

The Baby Skyrme model is special, as the restriction does have an intriguing additional symmetry. This can be important as there is not much difference of the solutions of the full model and the restriction, at least for the static case. We remind that the method can be easily extended to include the time dependence.

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